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SENSITIVITY ANALYSIS OF THIN-WALLED BARS

Summary

Sensitivity analysis of a three-dimensional elastic thin-walled structures being capable of incorporating large displacement and large rotation is developed and examined. The solution process and the formation of tangent operators are presented in a systematic manner and sensitivity for a response of system is formulated via direct differentiation method. A geometrically non-linear , three-dimensional rod model which incorporates transverse shear and torsion-warping deformation is developed. The model incorporates the classical notion bimoment and bi-shear in a geometrically exact context. The derived formulations are suitable for finite element implementation. An example which illustrates the performance of the formulation is presented.

1. Introduction

A thin-walled rod model established in this paper is generalization of the prismatic Timoshenko rod model through appending of warping displacement (Simo,Vu-Quoc (1991)), which makes possible to analyze such phenomena as lateral buckling and torsion instability. In this model it is not possible to analyze local instability (instability plates of profile).

Sensitivity analysis of systems has been presented by many authors , i.a.: Dems., Mróz (1985), Arora et al. (1988), Michaleris et al. (1994). An interesting application of the method can be found in optimization problems (Mróz et al. (1985)) and reliability analysis (Chodor, Bijak (1996)).

In this paper, a direct differentiation method for determining sensitivity of thin-walled structures is studied. We explore geometrically non-linear problems which are fundamental in the slender structures (e.g. engineering metal structures).

The central problem in sensitivity analysis is the determination of the implicit variations in the response fields generated by a specified design variation. In general, there are three classes of methods to solve this problem: the finite difference problem, the adjoin variable method and the direct differentiation method. Finite difference sensitivity analysis methods are simple to implement, but they can be computationally expensive and deficient in terms of accuracy and reliability (Michaleris et al. (1994)). For this reason, the adjoin variable and direct differentiation methods are generally preferred despite their relative complexity.

2. Computation of response of structure sensitivity

A steady-state non-linear problem may be expressed as:

$$\Psi(\delta\mathbf{a},\mathbf{a}) = \Psi_{\mathrm{int}}(\delta\mathbf{a},\mathbf{a}) - \Psi_{\mathrm{ext}}(\delta\mathbf{a},\mathbf{a}) = 0, \tag{1}$$

where: $\mathbf{a}$, $\delta\mathbf{a}$ - general displacement vector and its increment respectively, $\Psi_{\mathrm{int}}(\delta\mathbf{a},\mathbf{a})$ - internal force vector, $\Psi_{\mathrm{ext}}(\delta\mathbf{a},\mathbf{a})$ - external force vector.

In this paper, we shall investigate the response of system changes **a** due to changes in basic variables **b** (such as force, geometric and material parameters) by the use of a direct differentiation method.

2.1. Computation of the sensitivity by direct differentiation method in the incremental problem

Computation of the response of non-linear system with the standard incremental procedure is given by (Zienkiewicz, Taylor (1991)):

$$\mathbf{K}_T \cdot \delta\mathbf{a} = -\Psi(\mathbf{a}_I), \quad \mathbf{a}_{I+1} = \mathbf{a}_I + \delta\mathbf{a} \tag{2a,b}$$

where: $\mathbf{K}_T = \partial\Psi / \partial\mathbf{a}$ - tangent stiffness matrix, $\delta\mathbf{a}$ - increment of system response, Ψ - residual force corresponding to the level of response system $\mathbf{a}_I$, I-iteration step inside increment N. Relation (2) obtained for first-order Taylor series expansion (1) about point $\mathbf{a}_I$

Partial derivatives of residual forces (1), are evaluated by directly differentiation with respect to each design parameter b_k i.e.:

$$\mathbf{K}_T \cdot \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}b_k} = -\frac{\partial\Psi}{\partial b_k} \tag{3}$$

with the right side $\partial\Psi / \partial b_k$ called pseudo-load vector. In both equations (2a) and (3) tangent matrix $\mathbf{K}_T$ is the same.

2.2 Geometrically non-linear formulation

In geometrically non-linear problem right side vector in formula (3) is dependent on sensitivity in previous increment step *N-1* as follows (Michaleris et al. (1994)):

$$-\frac{\partial\Psi}{\partial b_k} = -\left(\frac{(\partial\Psi)_N}{(\partial\mathbf{a})_{N-1}} \cdot \frac{(\mathrm{d}\mathbf{a})_{N-1}}{\mathrm{d}b_k} + \frac{(\partial\Psi)_N}{\partial b_k}\right). \tag{4}$$

Derivative $(\mathrm{d}\mathbf{a})_{N-1} / \mathrm{d}b_k$ is known from previous step, and it is determined based on the derivative $(\mathrm{d}\mathbf{a})_{N-2} / db_k$ in advance step. We proceed in this way until start increment.

3. Thin-walled rod model

Position vectors describing the location of an arbitrary material point (X_1, X_2, S) in the initially straight thin-walled rod in the undeformed configuration $\mathbf{R}(X_\alpha, S)$ and in configuration after deformation $\mathbf{r}(X_\alpha, S)$ (α=1,2, $(X_1, X_2) \subset \Omega$, Fig. 1) is given by :

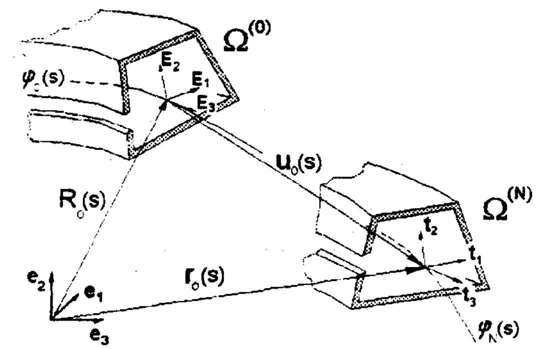

Fig. 1. Kinematic description of the thin-walled rod

$$\mathbf{R}(X_\alpha, S) = \mathbf{R}_o(S) + X_\alpha \mathbf{E}_\alpha \tag{5a}$$

$$\mathbf{r}(X_\alpha, S) = \mathbf{r}_o(S) + X_\alpha \mathbf{t}_\alpha + f(X_1, X_2) p(S) \mathbf{t}_3 \tag{5b}$$

where: $f(X_1 X_2)$ - is a prescribed (*given a priori*) warping function, and $p(S)$ - is the (unknown) warping amplitude. In the above equation, $\mathbf{r}_o$ describing position vector of the line of centroid and orthonormal basis $\mathbf{t}_i$ results from the rotation of the material (orthonormal) basis $\mathbf{E}_i$. Denoting the orthogonal transformation by $\Lambda = \mathbf{t}_i \otimes \mathbf{E}_i$, and inserting kinematic relation (5) into the definition of the deformation gradient tensor, the following expressions are derived:

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{r}}{\partial X_\alpha} \otimes \mathbf{E}_\alpha + \frac{\partial \mathbf{r}}{\partial S} \otimes \mathbf{E}_3 = \\ &= (\mathbf{t}_\alpha + f_{,\alpha} p \mathbf{t}_3) \otimes \mathbf{E}_\alpha + \left[(\mathbf{r}_o)' + \omega \times (X_\alpha \mathbf{t}_\alpha + fp\mathbf{t}_3) + fp'\mathbf{t}_3 \right] \otimes \mathbf{E}_3 \\ &= \Lambda \left\{ \mathbf{1}_3 + p\mathbf{E}_3 \otimes f_{,\alpha} \mathbf{E}_\alpha + \left[\Gamma + \mathbf{K} \times (X_\alpha \mathbf{E}_\alpha + fp\mathbf{E}_3) + fp'\mathbf{E}_3 \right] \otimes \mathbf{E}_3 \right\} \end{aligned} \tag{6}$$

In equation (6) $(\bullet)_{,\alpha} = \dfrac{\partial(\bullet)}{\partial X_\alpha}$, $(\bullet)' = \dfrac{\partial(\bullet)}{\partial S}$, and centroidal line strains are represented by vectors:

$$\Gamma = \Lambda^T (\mathbf{r}_o)' - \mathbf{E}_3, \quad \mathbf{K} = \Lambda^T \omega \tag{7a.b}$$

Beam curvatures are represented by the skew-symmetric tensor $(\hat{\cdot})$ or axial vector in spatial or material form, respectively:

$$\hat{\omega} = \omega \times = \frac{d\Lambda}{dS} \Lambda^T \qquad \hat{K} = \mathbf{K} \times = \Lambda^T \frac{d\Lambda}{dS} \tag{8a.b}$$

For the application of elastoplastic constitutive equation (actual development), it is proved convenient to introduce the second-order objective Biot strain tensor (Simo, Vu-Quoc (1991)):

$$\mathbf{H} = \Lambda^T \mathbf{F} - \mathbf{1}_3 = p\mathbf{E}_3 \otimes f_{,\alpha} \mathbf{E}_\alpha + \left[\Gamma + \mathbf{K} \times X_\alpha \mathbf{E}_\alpha + \mathbf{K} \times fp\mathbf{E}_3 + fp'\mathbf{E}_3 \right] \otimes \mathbf{E}_3 \tag{9}$$

Based upon the assumption of small deformation strains, but upon the arbitrary displacement and rotations, the Lagrangian strain tensor $\mathbf{E}$ is equal to corotational engineering strain tensor $\bar{\varepsilon}$:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{T} \cdot \mathbf{F} - \mathbf{1}_3\right) \cong \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{T}\right) = \bar{\varepsilon} \tag{10}$$

Invariant constitutive equation in terms of $\bar{\varepsilon}$ and its conjugate (in the sense of internal work) rotational Cauchy stress tensor $\bar{\sigma}$ can be written as:

$$\bar{\sigma} = \Lambda^{T}\sigma\Lambda = \mathbf{C} : \bar{\varepsilon} \cong \Lambda^{T}\mathbf{P} \tag{11}$$

where $\mathbf{P}$ represents first Piola-Kirchoff stress tensor and $\mathbf{C}$ is four-order modules tensor written in terms of the constant E and G for the isotropic elastic material.

The vectors of stress resultants: $\mathbf{n}$,$\mathbf{N}$, stress couples $\mathbf{m}$,$\mathbf{M}$ in spatial and material form respectively, bi-shear M_f and bi-moment B_f are obtained by the integration of stress vector over the cross-section:

$$\mathbf{n} = \int_A \mathbf{p}^3 dA \quad , \qquad \mathbf{N} = \Lambda^{T}\mathbf{n} = \int_A \bar{\sigma}_3 dA \, , \tag{12a,b}$$

$$\mathbf{m} = \int_A (\mathbf{r} - \mathbf{r}_0) \times \mathbf{p}^3 dA \quad , \qquad \mathbf{M} = \Lambda^{T}\mathbf{m} = \int_A \left[\Lambda^{T}(\mathbf{r} - \mathbf{r}_0)\right] \times \bar{\sigma}_3 dA \, , \tag{12c,d}$$

$$M_f = \mathbf{t_3} \cdot \int_A f_{,\alpha}\mathbf{p}^{\alpha} dA \qquad , \; B_f = \mathbf{E_3} \cdot \int_A f\bar{\sigma}_3 dA \; . \tag{12e,f}$$

where use has been made of the relation (Simo,Vu-Quoc (1991)):

$$\mathbf{p}^3 = \mathbf{P} \cdot \mathbf{E}_3 \qquad \bar{\sigma}_3 = \bar{\sigma} \cdot \mathbf{E}_3 = \Lambda^{T}\mathbf{p}^3 = \bar{\sigma}_{i3}\mathbf{E}_3 \tag{13a,b}$$

Relation (12) in vector form is expressed by:

$$\sigma_0 = \Pi\Sigma_0 . \tag{14}$$

where:

$$\Sigma_0 = \left[\mathbf{N}, \mathbf{M}, M_f, B_f\right]^{T}_{8x1} \quad \sigma_0 = \left[\mathbf{n}, \mathbf{m}, M_f, B_f\right]^{T}_{8x1}, \; \Pi = \begin{bmatrix} \Lambda & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_2 \end{bmatrix}_{8\cdot 8} \tag{15a-c}$$

For preparation of our linearization process, we explicitly derive the linearized constitutive equations resultant from form (12):

$$\begin{aligned} \Delta\mathbf{N} &= \int_A \mathbf{C}_3 \Delta\bar{\varepsilon}_3 dA \, , \\ \Delta\mathbf{M} &= \int_A \left[\Lambda^{T}(\mathbf{r} - \mathbf{r}_0)\right] \times \left[\mathbf{C}_3 \Delta\bar{\varepsilon}_3\right] dA \\ \Delta M_f &= \mathbf{t_3} \cdot \int_A f_{,\alpha}\mathbf{p}^{\alpha} dA \\ \Delta B_f &= \mathbf{E_3} \cdot \int_A f\mathbf{C}_3 \Delta\bar{\varepsilon}_3 dA \end{aligned} \tag{16a-d}$$

where $\mathbf{C}_3 = Diag(\mathrm{E,G,G})$ and change strain at arbitrary point $(X_1, X_2) \subset \Omega$ is given by:

$$\Delta\bar{\varepsilon}_3 = \mathbf{A_h} \cdot \Delta\varepsilon_0 . \tag{17}$$

$$\mathbf{A}_{\mathbf{h}} = \begin{bmatrix} 0 & 0 & 1 & X_2 & -X_1 & 0 & 0 & f \\ 1 & 0 & 0 & 0 & pf & -X_2 & f_{,1} & 0 \\ 0 & 1 & 0 & -pf & 0 & X_1 & f_{,2} & 0 \end{bmatrix}_{3\times 8}, \tag{18}$$

$$\Delta\boldsymbol{\varepsilon}_0 = \left[\Delta\Gamma, \Delta\mathbf{K}, \Delta p, \Delta p'\right]^T_{8\times 1}, \tag{19}$$

Virtual work expression (equilibrium equation) (1) for thin-walled rod is given by:

$$\Psi(\delta\mathbf{a},\mathbf{a}) = \int_{[0,L]} \left\{\Pi^T \mathbf{B}\delta\mathbf{a}\cdot\sigma_0 - \delta\mathbf{a}\cdot\mathbf{F}_{ext}\right\} dS = \Psi_{int}(\delta\mathbf{a},\mathbf{a}) - \Psi_{ext}(\delta\mathbf{a},\mathbf{a}) \tag{20}$$

where N_a, $\mathbf{B}_a$ represent the shape function and strain-displacement matrix, respectively:

$$\mathbf{B}_a^T = \begin{bmatrix} N_a'\mathbf{1}_3 & N_a\left[\hat{\mathbf{r}}_0'\right]^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N_a'\mathbf{1}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & N_a & N_a' \end{bmatrix}_{7\times 8} \tag{21}$$

Linearized finite element equation, is derived from (20) in form :

$$\mathbf{K}_T\Delta\mathbf{a} = \Psi_{int} - \Psi_{ext}, \tag{22}$$

where:

$$\mathbf{K}_T = \bigcup_e \left(\mathbf{K}_{ab}^{M} + \mathbf{K}_{ab}^{G}\right), . \tag{23}$$

In equation (22) $\Delta\mathbf{a} = \left[\Delta\mathbf{r}_0, \Delta\mathbf{w}, \Delta p\right]^T_{7x1}$ represent the incremental degrees of freedom of centroidal position vector $\mathbf{r}_o$, orthogonal transformation tensor Λ and warping amplitude p, respectively, $\mathbf{K}_{ab}^{M}$ represents the material stiffness matrix, $\mathbf{K}_{ab}^{G}$ represents geometric stiffness matrix, Ψ_{int}, Ψ_{ext} represent the internal and external force vector as follows:

$$\mathbf{K}_{ab}^{M} = \int_L \mathbf{B}_a^T\cdot\Pi^T\cdot\mathbf{D}\cdot\Pi\cdot\mathbf{B}_b dS, \quad \mathbf{D} = \int_A \mathbf{A}_{\mathbf{h}}^T\cdot\mathbf{C}\cdot\mathbf{A}_{\mathbf{h}} dA, \quad \mathbf{K}_{ab}^{G} = \int_L \mathbf{L}_a^T\cdot\mathbf{b}\cdot\mathbf{L}_a dS. \tag{24a-c}$$

In the above equation $\mathbf{L}_a$ and $\mathbf{b}$ represent displacement gradient matrix and stress matrix, respectively:

$$\mathbf{L}_a = \begin{bmatrix} N_a'\mathbf{1}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N_a'\mathbf{1}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_3 & \mathbf{0} \end{bmatrix}_{9\times 7}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\hat{\mathbf{n}} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\hat{\mathbf{m}} \\ \hat{\mathbf{n}} & \frac{1}{2}\hat{\mathbf{m}} & sym\left[\mathbf{n}\otimes\mathbf{r}_0' - \left(\mathbf{r}_0'\cdot\mathbf{n}\right)\mathbf{1}_3\right] \end{bmatrix}_{9\times 9} \tag{25}$$

4. Shape sensitivity analysis of thin-walled rod

Consider a finite element discretization of the reference line of centroids φ_0 (Fig.1) consisting of N_{node} nodal points $\mathbf{R}_0^I$ I=1,..., N_{node}. Denoting by $N_I(\xi)$ the N^e_{node} element shape functions with isoparametric co-ordinate in the element $\xi\in[-1,1]$. The mapping

$$\xi\in[-1,1]\rightarrow \mathbf{R}_0(\xi)\big|_L = \sum_{I=1}^{N^e_{node}} N_I(\xi)\,\mathbf{R}_0^I, \tag{26}$$

then defines a parametrization of φ_0 via local element shape functions.

The local parametrization (26) of the curve φ_0 is related to the arc-length parametrization used in the preceding section via the local differential relation:

$$dS = \left|\mathbf{J}^h(\xi)\right| \cdot d\xi \quad \text{where} \quad \mathbf{J}_e^h(\xi) = \sum_{I=1}^{Nen} \frac{dN_I(\xi)}{d\xi} \cdot \mathbf{R}_0^I \tag{27a,b}$$

called the jacobian isoparametric transformation.
As an example, for linear isoparametric elements $N^e_{node} = 2$ while $N_1(\xi) = \frac{1}{2}(1-\xi)$ and $N_2(\xi) = \frac{1}{2}(1+\xi)$ are the local element shape functions (Zienkiewicz, Taylor (1989)):

Notice that the reduced numerical integration should be used in (24) to avoid serious shear locking for low-order interpolations or membrane locking for higher-order interpolations. No spurious modes are known to appear as a result of reduced integration for beam element. (As an example possible schemes are (a) linear, isoparametric interpolation functions-reduced quadrature : 1 point Gauss quadrature, full quadrature: 2 Gauss quadrature, (b) qudratic, isoparametric interpolation functions- reduced quadrature : 2 point Gauss quadrature, full quadrature: 3 Gauss quadrature).

The sensitivities of an integral expression are computed after its transformations into the isoparametric domain (Vidal, Haber (1993)), whose shape does not depend on the design variables. The jacobian of this transformation $\left|\mathbf{J}^h(\xi)\right|$ can be expressed in terms of the nodal coordinates, so that, it can also be differentiated in order to known the integral sensitivities. Using this techniques right side pseudo-load vector ($\partial\Psi / \partial b_k$), computation of the response of system sensitivity (formulas (3,4)) is given by

$$\frac{\partial \Psi_{int}(\delta\mathbf{a},\mathbf{a})}{\partial b_k} = \frac{\partial \int\limits_{[-1,1]} \Pi^T \mathbf{B}\delta\mathbf{a} \cdot \sigma_0 \mathbf{J}\, d\xi}{\partial b_k} = \tag{28}$$

$$= \int\limits_{[-1,1]} \left\{ \frac{\partial \Pi^T}{\partial b_k} \mathbf{B} \cdot \sigma_0 \mathbf{J} + \Pi^T \frac{\partial \mathbf{B}}{\partial b_k} \cdot \sigma_0 \mathbf{J} + \Pi^T \mathbf{B} \frac{\partial \sigma_0}{\partial b_k} \cdot \mathbf{J} + \Pi^T \cdot \mathbf{B} \cdot \sigma_0 \cdot \frac{\partial \mathbf{J}}{\partial b_k} \right\} d\xi$$

where the sensitivity of the jacobian is:

$$\frac{\partial \mathbf{J}}{\partial b_k} = \mathbf{J}\, tr\left(\mathbf{J}^{-1} \frac{\partial \mathbf{J}}{\partial b_k} \right) \tag{29}$$

Because this paper is an incroduction to sensitivity analysis of the elasto-plastic systems, we make numerically integration over cross-section by Gauss quadrature. We assume that cross-section consist of elements of equal thickness, location of an arbitrary point $\mathbf{X} = X_\alpha \mathbf{E}_\alpha$ of cross-section is given by isoparametric interpolation:

$$\xi, \eta \in [-1,1] \rightarrow \mathbf{X} = \sum_{I=1}^{N^e_{node}} N_I(\xi)\, \mathbf{X}_0^I + \eta \cdot \frac{t}{2} \cdot \mathbf{v}(\xi), \tag{30}$$

where $\mathbf{X}_0^I$ is position nodal point on the middle-line in cross-section element, t is thickness of section element, $\mathbf{v}(\xi)$ is vector perpendicular to middle-line.

5. Example

Thin-walled beam is made of material described by Young module E=205GP and Kirchoff module G=80 GPa. Beam by length l=6 m is made of a I-section : cross-section height h=300 mm, flange thickness t_f=10.7 mm, web thickness t_w=7.1 mm. It is subjected a concentrated external torsional moment M_s=2kNm on free end.

The relationships between the sensitivity of free end rotation ϕ on the changes of flange width b and nominal width b are shown in Fig. 2.

Sensitivity of rotation on the changes of flange width is relationally big in the whole examined range and rises with decreasing nominal flange width. For I-section IPE300 nominal flange width is 150 mm and senistivity of rotation angle is about 15-times less than sensitivity for rectangular cross-section.

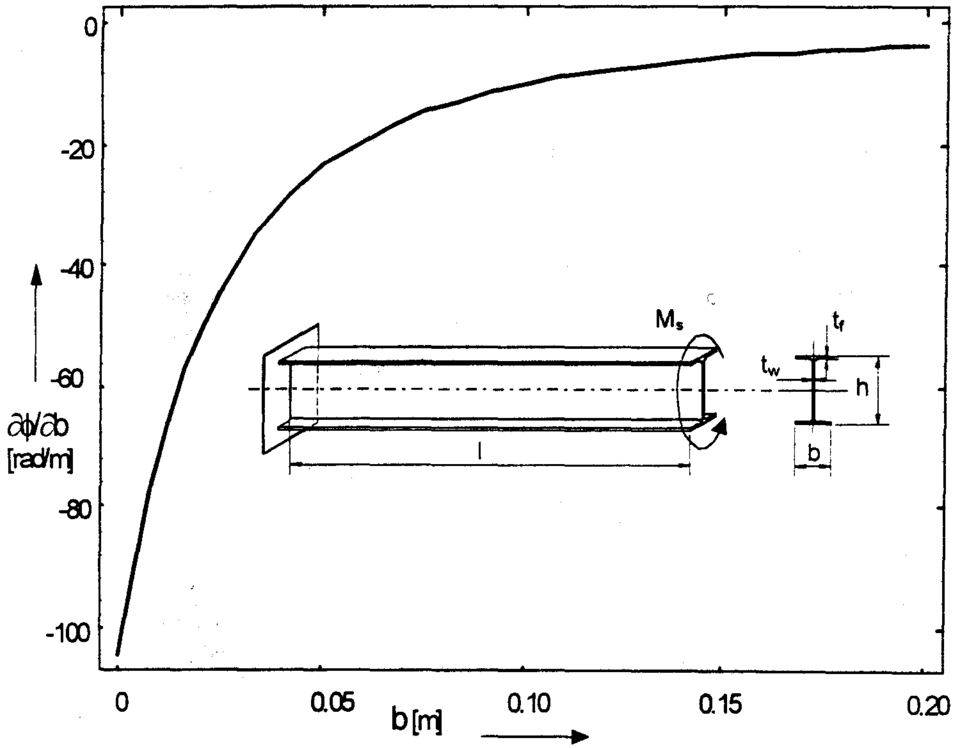

Fig. 2. Sensitivity of thin-walled beam free end rotation on the changes width flange in relationship of nominal width b

6. Conclusion

1. Computation sensitivity of the geometrically non-linear system response of 3-D structures consisting of thin-walled rods is effected parallely with standard incremental procedure.

2. In this paper sensitivity for a response of system is formulated via direct differentiation method due to possibility of its simple extension to the elastic-plastic range. Such procedures are currently being developed by the present authors.

3. Direct differentiation method requires analytical calculations of derivative expression of matrix occurring in residue force.

4. Isoparametric formulation make possible analysis of response sensitivity to change in structure shape.

5. The model of a thin-walled rod formulated with use of strain as a symmetric part of Biot tensor make possible a relatively simple extension of sensitivity analysis to the case initial curvatures and pretwists of a thin-walled rod.

6. The adopted measure of strain leads to simple expressions of residual force, which simplifies calculations of derivatives in the method direct differentiation, whereas Green-Lagrange strain measure, would lead to complicated calculations.

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