Sensitivity of Cross-Section Shape for Nonlinear Thin-Walled Bars

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ABSTRACT: Sensitivity analysis of a cross-section shape for three-dimensional elastic thin-walled rod structures being capable of incorporating large displacement and large rotation is developed and examined. A thin-walled rod model is generalisation of the prismatic Timoshenko rod model through appending of warping displacement, which makes possible to analyse such phenomena as lateral buckling and torsion instability. The model incorporates the classical notion of bi-moment and bi-shear in a geometrically exact context. The warping function *f* is derived from differential equation end boundary condition for the Saint-Venant uniform torsion problem for prismatic body using Galerkin method. The warping function and the cross-section shape are approximated by the eight-node isoparametric element. The shape of cross-section rod is described by Bezier representation.

1. INTRODUCTION

A sensitivity analysis of systems has been presented by many authors i.a.: Dems., Mróz (1985), Arora et al. (1988). The central problem in sensitivity analysis is the determination of the implicit variations in the response fields generated by a specified design variation. In this paper, we shall investigate the response of system changes ***a*** due to changes in cross-section shape parameters ***b*** by the use of a direct differentiation method.

A thin-walled rod model established here is generalisation of the rod model Simo,Vu-Quoc (1991). This model incorporates the classical notion of bi-moment and bi-shear in a geometrically exact context.

The warping function is derived from differential equation end boundary condition for the Saint-Venant uniform torsion problem for prismatic body using Galerkin method (Schramm, Pilkey (1994)). The sensitivities of an integral expression are computed after its transformations into the isoparametric domain, whose shape does not depend on the design variables. The Jacobian of this transformation can be expressed in terms of the nodal coordinates, so that, it can also be differentiated in order to know the integral sensitivities.

2. MODEL OF THIN-WALLED ROD

In Fig.1, $\mathbf{r}_o$ denoting position vectors describing the line of centroid and orthonormal basis $\mathbf{t}_i$ results from the rotation of the material orthonormal basis $\mathbf{E}_i$ by the orthogonal transformation $\Lambda = \mathbf{t}_i \otimes \mathbf{E}_i$. Location of an arbitrary material point (X_1, X_2, S) in the thin-walled rod in the undeformed configuration $\mathbf{R}(X_\alpha, S)$ and in configuration after deformation $\mathbf{r}(X_\alpha, S)$ (α=1,2 , $(X_1, X_2) \subset \Omega$ and S- arc length parameter of reference line of centroids, see Fig. 1) is given by:

$$\mathbf{R}(X_\alpha, S) = \mathbf{R}_o(S) + X_\alpha \mathbf{E}_\alpha \tag{1a}$$

$$\mathbf{r}(X_\alpha, S) = \mathbf{r}_o(S) + X_\alpha \mathbf{t}_\alpha + f(X_1, X_2)p(S)\mathbf{t}_3 \tag{1b}$$

where $p(S)$ - the unknown warping amplitude and $f(X_1, X_2)$ - a prescribed warping function, which for an arbitrary cross-section of a prismatic beam are found from Laplace equation (Simo, Vu-Quoc(1991), Schramm, Pilkey (1994)):

$$\Delta f = 0\ ,\ \text{in} A \tag{2}$$

and a boundary condition

$$\partial f / \partial \bar{\mathbf{n}} = X_2 \bar{n}_1 - X_1 \bar{n}_2 \text{ on } \partial A. \tag{3}$$

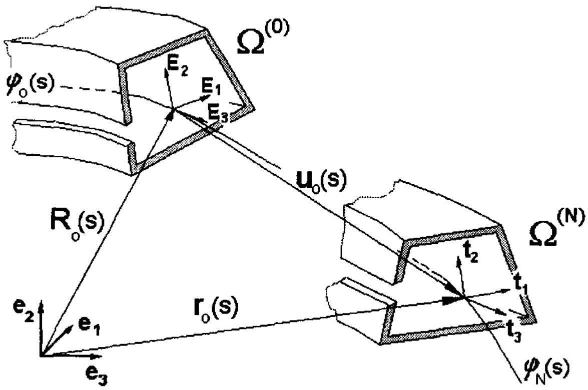

Fig. 1. Kinematic description of the thin-walled rod

Here, the vector $\bar{\mathbf{n}} = \{\bar{n}_1, \bar{n}_2\}$ is the normal unit vector on the boundary ∂A of cross-section A and

$$(\bullet)_{,\alpha} = \partial(\bullet)/\partial X_\alpha\ .$$

Using the isoparametric formulation of finite element analysis, the linearized incremental equilibrium equation is discretized for numerical calculation in the form:

$$\mathbf{K}_T \Delta\mathbf{a} = -\Psi(\mathbf{a}_I) \tag{4a}$$

$$\mathbf{a}_{I+1} = \mathbf{a}_I + \Delta\mathbf{a} \tag{4b}$$

where

$$\mathbf{K}_T = \bigcup_e \left(\mathbf{K}_{ab}^{\mathrm{M}} + \mathbf{K}_{ab}^{\mathrm{G}}\right), \quad \Psi = \bigcup_e \left(\Psi_{int(a)} - \Psi_{ext(a)}\right). \tag{5a,b}$$

In equation (5) $\mathbf{K}_{ab}^{\mathrm{M}}$, $\mathbf{K}_{ab}^{\mathrm{G}}$ represents the material and geometric stiffness matrix respectively and $\Psi_{int(a)}, \Psi_{ext(a)}$ the internal and external force vector as follows:

$$\mathbf{K}_{ab}^{\mathrm{M}} = \int_{\mathrm{L}} \mathbf{B}_a^T \cdot \Pi \cdot \mathbf{D} \cdot \Pi^T \cdot \mathbf{B}_b dS^h\ ,\quad \mathbf{K}_{ab}^{\mathrm{G}} = \int_{\mathrm{L}} \mathbf{L}_a^T \cdot \mathbf{b} \cdot \mathbf{L}_b dS^h\ ,$$

$$\Psi_{int(a)} = \int_{\mathrm{L}} \mathbf{B}_a^T \cdot \sigma_0 dS^h\ ,\quad \Psi_{ext(a)} = \int_{\mathrm{L}} \left[N_a \bar{\mathbf{n}}, N_a \bar{\mathbf{m}}, N_a \bar{B}_f\right]^T dS^h \tag{6a-d}$$

where $\bigcup_e$ is finite element assemblages, N_a shape functions and $\mathbf{B}_a$ represents the strain-displacement matrix

$$\mathbf{B}_a^T = \begin{bmatrix} N_a' \mathbf{1}_3 & 0 & 0 & 0 \\ -N_a\left[\hat{\mathbf{r}}_0'\right] & N_a' \mathbf{1}_3 & 0 & 0 \\ 0 & 0 & N_a & N_a' \end{bmatrix}_{7\times 8} \quad \text{and } \Pi = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \mathbf{1}_2 \end{bmatrix}_{8\times 8}\ . \tag{7a,b}$$

Here $\mathbf{L}_a$ and $\mathbf{b}$ represent displacement gradient matrix and stress matrix, respectively:

$$\mathbf{L}_a = \begin{bmatrix} N_a' \mathbf{1}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N_a' \mathbf{1}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_3 & \mathbf{0} \end{bmatrix}_{9\times 7}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\hat{\mathbf{n}} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\hat{\mathbf{m}} \\ \hat{\mathbf{n}} & \frac{1}{2}\hat{\mathbf{m}} & \hat{\mathbf{r}}' \cdot \hat{\mathbf{n}} \end{bmatrix}_{9\times 9} \quad \text{and}$$

$$\mathbf{D} = \int_A \begin{bmatrix} G & \bullet & \bullet & \bullet & \bullet & -GX_2 & Gf_{,1} & \bullet \\ & G & \bullet & \bullet & \bullet & GX_1 & Gf_{,2} & \bullet \\ & & E & EX_2 & -EX_1 & \bullet & \bullet & Ef \\ & & & EX_2^2 & -EX_1X_2 & \bullet & \bullet & EfX_2 \\ & & & & EX_1^2 & \bullet & \bullet & -EfX_1 \\ & s & y & m & & C_1 & C_2 & \bullet \\ & & & & & & C_3 & \bullet \\ & & & & & & & Ef^2 \end{bmatrix} dA \tag{8a-c}$$

where

$$C_1 = G\left(X_1^2 + X_2^2\right), \; C_2 = G\left(-f_{,1}X_2 + f_{,2}X_1\right), \; C_3 = G\left(f_{,1}^2 + f_{,2}^2\right). \tag{9a-c}$$

In equation (4) $\Delta\mathbf{a} = [\Delta\mathbf{r}_0, \Delta\mathbf{w}, \Delta p]_{7x1}^T$ represent the incremental degrees of freedom of centroidal position vector $\mathbf{r}_o$, orthogonal transformation tensor Λ and warping amplitude p, respectively, and $\boldsymbol{\sigma}_0 = \left[\mathbf{n}, \mathbf{m}, M_f, B_f\right]_{8x1}^T$ is generalized stress vector. The vectors of stress resultans $\mathbf{n}$, stress couples $\mathbf{m}$, bi-shear M_f and bi-moment B_f are obtained by the integration of stress vector related to first Piola-Kirchoff stress tensor over the cross-section. The distributed applied force $\overline{\mathbf{n}}$, couple $\overline{\mathbf{m}}$ and bi-moment $\overline{B}_f$ are given in resultant force vector $\overline{\boldsymbol{\sigma}}_0 = \left[\overline{\mathbf{n}}, \overline{\mathbf{m}}, \overline{B}_f\right]_{7x1}^T$. In equations (7,8) $\hat{\mathbf{a}}$ denote skew-symmetric matrix of axial vector $\mathbf{a}$.

3. SENSITIVITY ANALYSIS OF CROSS-SECTION SHAPE

In this paper, we shall investigate the response of system changes $\mathbf{a}$ due to changes in cross-section shape parameters $\mathbf{b}$ by the use of a direct differentiation method. The sensitivity of displacement respect to each design parameter b_k is given by:

$$\mathbf{K}_T\left\{\partial(\Delta\mathbf{a})/\partial b_k\right\} = -\,\partial\Psi\left(\mathbf{a}_I\right)/\partial b_k \tag{10a}$$

where so called "quasi-load" vector:

$$\partial\Psi\left(\mathbf{a}_I\right)/\partial b_k = \bigcup_e \partial\left(\int_L \mathbf{B}_a^T\left(\mathbf{a}_I\right)\cdot\boldsymbol{\sigma}_0\left(\mathbf{a}_I\right) dS^h\right)\Big/\partial b_k \tag{10b}$$

will be obtained analytically or numerically by "exact" differentiation (Olhoff et all (1993)). In both formulas (4) and (10a) tangent stiffness matrix $\mathbf{K}_T$ appears.

In the case of the sensitivity response of the geometrically non-linear problem in increment step *N* right side vector in formulae (10a) is dependent on sensitivity in previous increment step *N-1* as follows (Michaleris et al. (1994)):

$$-\partial\Psi_{int}/\partial b_k = -\left\{ (\partial\Psi_{int})_N/(\partial\mathbf{a})_{N-1}\cdot(\mathbf{da})_{N-1}/\partial b_k + (\partial\Psi_{int})_N/\partial b_k \right\}, \tag{10c}$$

Derivative $(\mathbf{da})_{N-1}/\mathrm{d}b_k$ is known from previous step, and it is determined based on the derivative $(\mathbf{da})_{N-2}/db_k$ in advance step. We progress so until start increment.

We assume that the cross-section consists of a curve segment. A set of co-ordinate vectors $\mathbf{p}_1, \mathbf{p}_2, \dots \mathbf{p}_n$ of n points $P_1, P_2, \dots P_n$ named "control points" defined current position vector $\mathbf{X} = \{X_1, X_2\}$ on the cross-section curve segment from Bezier parametrization:

$$\mathbf{X}(t) = \sum_{k=0}^{n} B_{kn}(t)\cdot\mathbf{p}_k \ . \tag{11}$$

The weighting function $B_{kn}(t)$ is given by $B_{kn}(t) = C_n^k t^k (1-t)^{n-k}$ where the factors $C_n^k = \binom{n}{k}$ are the binomial coefficients and parameter $t \in (0,1)$. The change of the vertex co-ordinates $\mathbf{p}_k$ is used to control the cross-sectional shape.

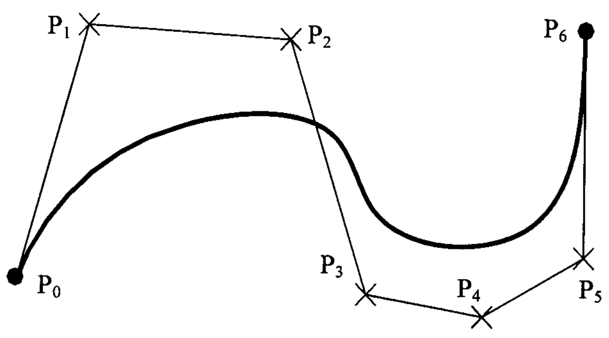

Fig. 2. Examples of Bezier parametrization

The finite element formulation in order to calculate torsional characteristics can be derived from the differential equation (2) and boundary condition (3) using Galerkin method (Schramm, Pilkey (1994)). The warping function $f(X_1, X_2)$ will be approximated elementwise by $f(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{f}^e$. Here $\mathbf{f}^e$ is the vector of the unknown values of the warping function and $\mathbf{N}(\xi,\eta)$ is the matrix of shape function in mathematical co-ordinates $\xi, \eta \in [-1,1]$.

The element equation appears as (Schramm, Pilkey (1994)):

$$\mathbf{k}^e \cdot \mathbf{f}^e = \mathbf{F}^e, \tag{12}$$

including the element "stiffness" matrix of

$$\mathbf{k}^e = \int_{-1}^{1}\int_{-1}^{1} (\tilde{\nabla}\mathbf{N})^{\mathrm{T}} (\mathbf{J}^{-1})^{\mathrm{T}} \mathbf{J}^{-1} (\tilde{\nabla}\mathbf{N}) \det \mathbf{J}\, d\xi d\eta\, , \tag{13}$$

and the element "load" vector of

$$\mathbf{F}^{\mathbf{e}} = \int_{-1}^{1}\int_{-1}^{1}\left(\tilde{\nabla}\mathbf{N}\right)^{\mathrm{T}}\left(\mathbf{J}^{-1}\right)^{\mathrm{T}}\begin{bmatrix} -X_2 \\ X_1 \end{bmatrix}\det \mathbf{J}\, d\xi\, d\eta\,, \tag{14}$$

The operator $\tilde{\nabla}$ is given by $\tilde{\nabla} = \{\partial/\partial\xi, \partial/\partial\eta\}$. The quantity $\det \mathbf{J}$ is the determinant of the Jacobian which follows from the mapping of the finite element. The mapping functions describe the element geometry appears as

$$\mathbf{X}(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{X}^e\,, \tag{15}$$

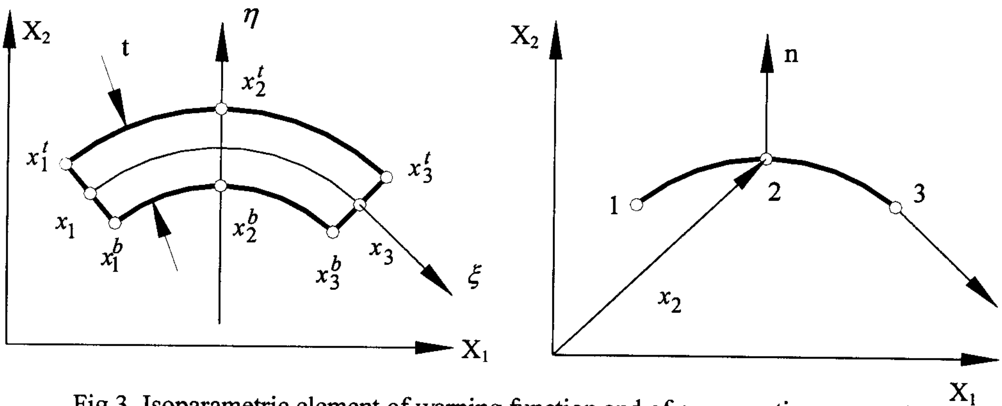

Fig.3. Isoparametric element of warping function and of cross-section segment

where $\mathbf{X}^e$ is the vector of the nodal co-ordinates of *eight-node element* (Fig 3a). In the case of constant thickness, location of an arbitrary point of cross-section reduced to interpolation:

$$\mathbf{X}(\xi,\eta) = \sum_{i=1}^{3} N_i(\xi)\left[\mathbf{X}\{t(i)\} + \eta\mathbf{n}\{t(i)\}\cdot\frac{g}{2}\right],\,, \tag{16}$$

where $\mathbf{X}\{t(i)\}, \mathbf{n}\{t(i)\}$ is position nodal point number *i* on the middle-line in cross-section element and vector perpendicular to middle-line respectively, g is thickness of section element (Fig 3b). The terms $N_i(\xi)$, (i= 1,2,3) are quadratic shape functions. The nodal parametric co-ordinates $t(i)$ of the line element can be obtained from

$$t(i) = \frac{2\tau + i - 3}{2\tau_{\max}}, \quad \tau = 1,2,\ldots\tau_{\max}\,,\; i{=}1,2,3 \tag{17}$$

where τ is the member of the finite element defined by the Bezier curve, *i* is the node number related to the line element τ and $\tau_{\max}$ is total number of finite elements in curve.

From equation (16) the Jacobian $\mathbf{J}$ is given by:

$$\mathbf{J}(\xi,\eta) = \sum_{i=1}^{3}\begin{bmatrix} \dfrac{\partial N_i}{\partial\xi}\left[\mathbf{X}\{t(i)\} + \eta\mathbf{n}\{t(i)\}\cdot\dfrac{g}{2}\right]^{\mathrm{T}} \\ N_i(\xi)\left[\mathbf{n}\{t(i)\}\right]^{\mathrm{T}}\cdot\dfrac{g}{2} \end{bmatrix}. \tag{18}$$

Consideration of the compatibility of the warping function leads to linear system equation which has to be solved to obtain the nodal values of the warping function.

Differentiation of the element relationship of equation (12) leads to :

$$\mathbf{k}^e \cdot \frac{\partial \mathbf{f}^e}{\partial b_k} = \frac{\partial \mathbf{F}^e}{\partial b_k} - \frac{\partial \mathbf{k}^e}{\partial b_k} \mathbf{f}^e . \tag{19}$$

Explicit form of the derivatives of the element „stiffness" matrix $\mathbf{k}^e$ and element "load" vector are found in Schramm, Pilkey (1994).

4. EXAMPLE

In the following example we shall determine sensitivity of twisting angle of a beam due to changes of flanges flexion of its cross-section measured with **b** parameter. Steel cantilever beam is 2000mm long and has I-section similar to standard HEA240 but with no fillets.

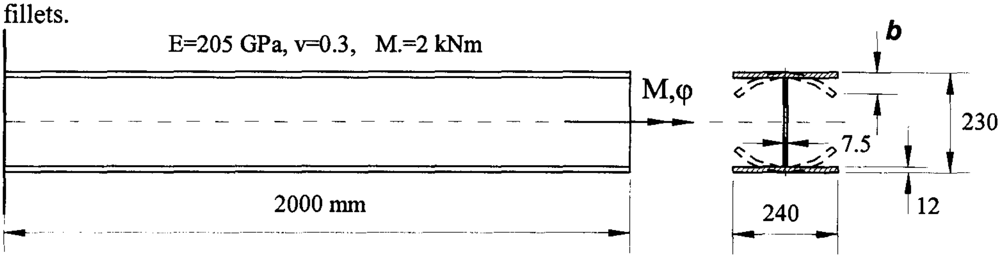

Fig. 4. Exemplary beam

The whole problem can be divided into two stages:
1) analysis of cross-section of the beam with finite elements discretization in order to determine the warping function,
2) static analysis with finite beam elements discretization.

We pay particular attention to stage 1). Procedure of solving geometrical nonlinear problem in stage 2) was described by Chodor and Bijak (1996). The beam in the example was divided into 8 finite elements longwise.

Notice that geometrical nonlinearity of structure is of importance only in second stage of calculations. However, analysis of cross-section, i.e. torsion equations (2), is linear.

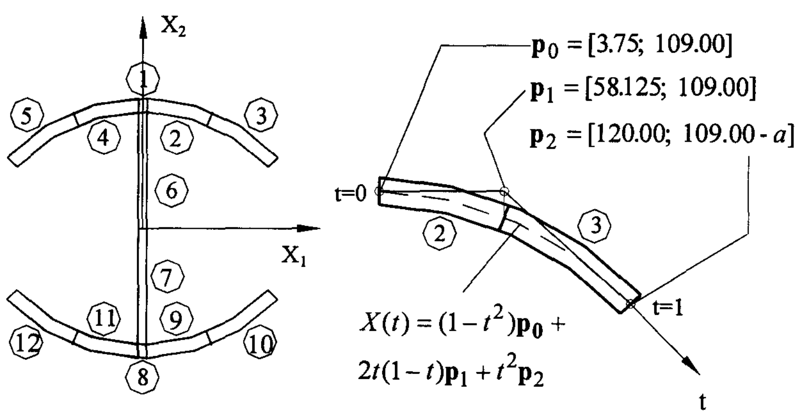

Fig. 5. Cross-section of exemplary beam

The cross-section of the beam was divided into 12 finite elements (Fig. 5). For curved flanges we used approximation with second-order Bezier function.
Shape functions of three nodes on element axis for calculation of Jacobian are equal to: $N_1 = -1/2\xi(1-\xi)$, $N_2 = 1-\xi^2$, $N_3 = 1/2\xi(1+\xi)$, and shape functions for eight-node element were published by Zienkiewicz, Taylors (1989/1991).
Let's consider, for example values of fundamental variables in solution of the problem for element (3) and for $\underline{b = 0, \xi = 0, \eta = 0}$:

- coordinates of nodes for calculation of Jacobian
  X(t = 0.5) = [60.00;109,00], X(t = 0.75) = [89.5312;109,00], X(t = 1.00) = [120,00;109,00]
  n(t = 0.5) = n(t = 0.75) = n(t = 0.75) = [0;1],
- Jacobian $\mathbf{J} = \begin{bmatrix} 30.00 & 0 \\ 0 & 6.00 \end{bmatrix}$,
- derivatives from Jacobian and derivative of Jacobian determinant
  $$\partial\mathbf{J}/\partial b = \begin{bmatrix} 0 & -0.375 \\ 0.075 & 0 \end{bmatrix}, \quad \partial Det[\mathbf{J}]/\partial b = 0\,.$$
- derivatives of coordinates of section in Gaussian integration points (we assumed 3 integration points) $\partial X/\partial b = \begin{bmatrix} 0 & -0.5625 \end{bmatrix}$.

After calculation of $\mathbf{k}^e$ matrix and $\mathbf{F}^e$ vector for element (e) according to equations (13), (14) we need to assemble global matrices in a standard FEM way.

Simultaneously with stiffness matrix assembling procedure for calculation of warping functions we put together derivatives which are found in equation (19) in order to calculate derivatives of warping functions.

Generated linear equations (12) and (19) are solved using the same stiffness matrix $\bigcup_e \mathbf{k}^e$.

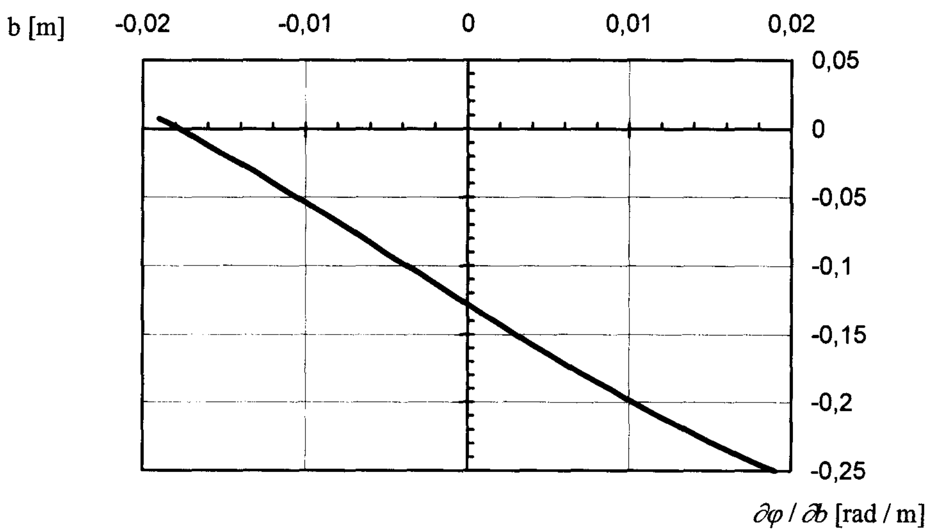

Fig.6. Sensitivity of torsion angle of the beam to imperfection **b** of shape of its cross-section

On the ground of the calculated values of the warping function and its derivatives we calculate tangent stiffness matrix **D**. With the rule: $\partial \mathbf{J}^{-1} / \partial b_k = -\mathbf{J}^{-1} \cdot \partial \mathbf{J} / \partial b_k \cdot \mathbf{J}^{-1}$ derivative of **D** was obtained and next with use of equation (10) required sensitivity $\partial\varphi / \partial b$ of the torsion angle could be calculated.

Fig. 6 shows dependence of sensitivity $\partial\varphi / \partial b$ on nominal imperfection **b** of cross-section's shape. The absolute value of beam's sensitivity to positive imperfections **b** (deflection of the flanges towards inside of the cross-section) is greater than sensitivity on negative imperfections (deflections outside the section). Flow of the sensitivity $\partial\varphi / \partial b$ is practically linear in the whole investigated range.

5. CONCLUSIONS

In this work an analysis we have shown of sensitivity of geometrical nonlinear, elastic thin-walled beams to changes of cross-section's shape. The change can be arbitrary and it may concern changes of wall thickness or changes of section shape (initial deflections of walls).

The warping function was obtained from solution of differential equation and boundary conditions for uniform torsion problem for prismatic body using Galerkin method. The warping function and cross-section shape are approximated by the eight-node isoparametric element. The shape of cross-section rod is described by Bezier representation.

It was shown that solution of warping equation is a linear problem and geometrical nonlinearities are of importance only at the stage of solving the beam similarly to problems in which changes of cross-section's shape are not considered.

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